

2. THEORY AND COMPUTER PROGRAM

The problem of ground-wave propagation has a long history and has been covered extensively in the literature. A rather complete list of this literature can be found in a paper by Wait (1964), which includes a very good discussion of the problem.

To make the results given in this report more meaningful, an outline of the method of solution will be presented. It is meant only as a heuristic discussion and not as a rigorous treatment; quite interesting and rigorous treatments can be found in papers by Hufford (1952), Wait (1964), Ott and Berry (1970), and Ott (1971a, b).

We begin by considering Maxwell's equations. It can be shown (Stratton, 1941, ch. 1) for a linear isotropic medium of permittivity ϵ , permeability μ , and conductivity σ , that Maxwell's equations can be expressed in terms of the Hertzian potential $\vec{\Pi}$, which must satisfy the Helmholtz equation

$$\nabla^2 \vec{\Pi} + k^2 \vec{\Pi} = 0 \quad (1)$$

at all ordinary points in space, and which has the following relation to the \vec{E} and \vec{H} fields:

$$\vec{H} = (1/\mu) \vec{\nabla} \times (i\omega\mu\epsilon \vec{\Pi}), \quad (2)$$

$$\vec{E} = \vec{\nabla} (\vec{\nabla} \cdot \vec{\Pi}) + \omega^2 \mu \epsilon \vec{\Pi}. \quad (3)$$

In (1), k is the complex wave number

$$k^2 = k_0^2 \mu_r \epsilon_r, \quad (4)$$

where

$$\epsilon_r = \epsilon_r - i\sigma/\omega\epsilon_0, \quad \mu_r = \mu/\mu_0, \quad \epsilon_r = \epsilon/\epsilon_0, \quad (5)$$

and where μ_0 and ϵ_0 are the free-space values of μ and ϵ ; k_0 is the free-space wave number (ω/c); ω is the angular wave frequency; c is the velocity of light; an ordinary point is one in whose neighborhood the physical and electrical properties of the medium vary in a smooth and continuous manner; and $\vec{\Pi}$ is assumed to contain an implicit time dependence of $\exp(i\omega t)$. Since the transmitting and receiving antennas and the earth's surface mark points where the properties of the medium change abruptly, they are not ordinary points and, hence, must be excluded from the space where (1) is valid. This space is contained within the surface S shown in figure 2. This surface consists of a half-sphere at infinity S_∞ , the earth's surface S_e , and the small sphere and half-sphere around the singular points. The antennas are assumed to be short vertical electric dipoles, in which case $\vec{\Pi}$ for the radiation field in the vicinity of the transmitting antenna takes the particularly simple form (Stratton, 1941, sec. 8.4 - 8.7)

$$\vec{\Pi}(W \text{ near } T) \cong \Pi_0 \frac{\exp(-ik_0|\vec{d}|)}{|\vec{d}|} \hat{z}, \quad (6)$$

where

$$\Pi_0 = -\frac{i I_0 \ell}{4\pi \omega \epsilon_0}, \quad (7)$$

and where I_0 is the amplitude of the current in the antenna, ℓ is the effective length of the antenna, and \hat{z} is a unit vector in the z -direction. Using the above assumption, applying Green's theorem (Morse, 1953, sec. 7.2) to (1), and using the fact that the integral vanishes on S_∞ , we obtain the following form for the z -component, Π_z , of $\vec{\Pi}$ in the vicinity of the receiver:

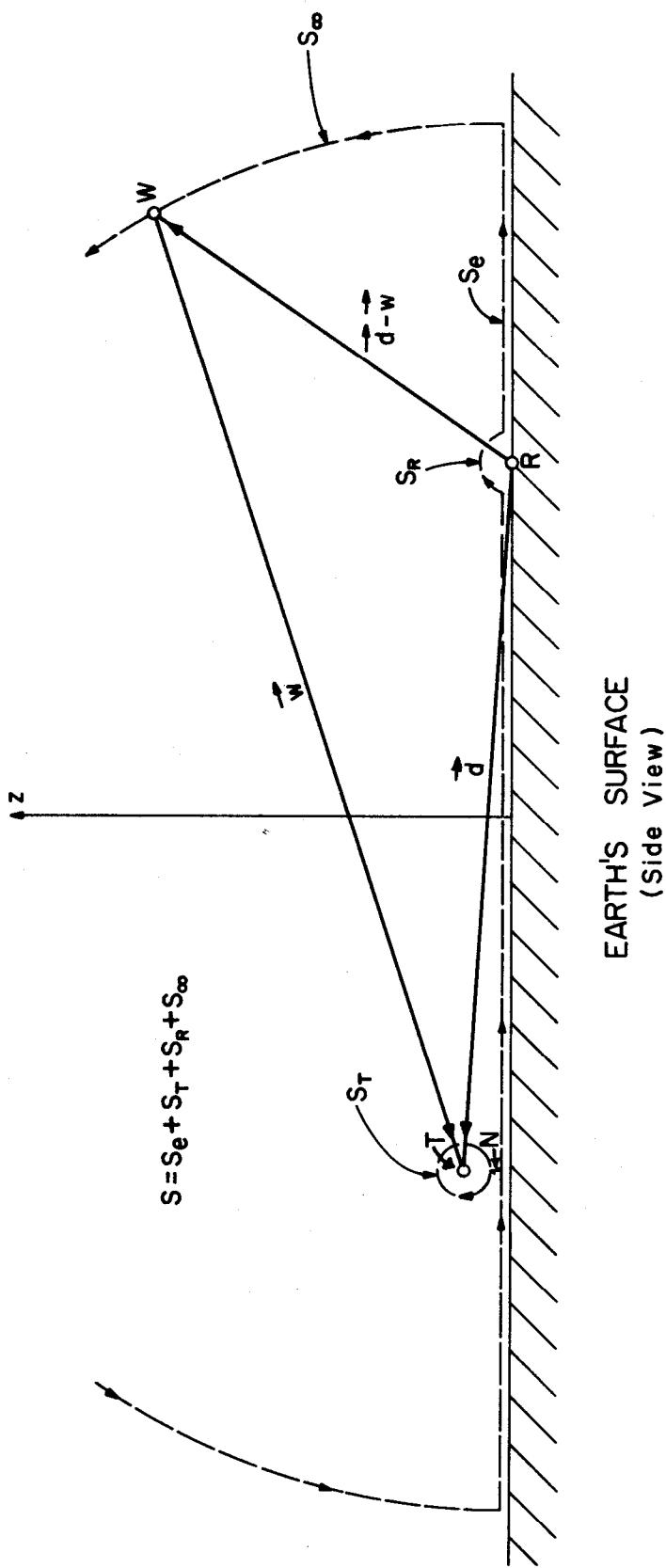


Figure 2. Side view of the path geometry and surfaces of integration.

$$\begin{aligned}
 \Pi(R) = & 2 \Pi_0 \frac{\exp(-ik_0 |\vec{d}|)}{|\vec{d}|} \\
 & + \frac{1}{2\pi} \int_{S_e} \int \left[\Pi(W) \frac{\partial}{\partial z} \cdot \left(\frac{\exp(-ik_0 |\vec{w}|)}{|\vec{w}|} \right) \right. \\
 & \left. - \frac{\exp(-ik_0 |\vec{w}|)}{|\vec{w}|} \cdot \frac{\partial \Pi(W)}{\partial z} \right] dS. \quad (8)
 \end{aligned}$$

All other components of $\vec{\Pi}$ are zero. If we now further assume that the ratio of $(\partial \Pi / \partial z)$ to Π depends only upon the physical and electrical properties of the earth's surface at each point, that is, assume that the concept of surface impedance is valid (Godzinski, 1961), then we have the following relation (Bremmer, 1954; King, 1965; and others),

$$\frac{\partial \Pi(W)}{\partial z} \approx (ik_0 / \eta_0) Z(W) \Pi(W) \quad (9)$$

at the earth's surface, where

$$Z(W) = \eta_0 / \sqrt{\epsilon_c(W)}. \quad (10)$$

Equation (8) then takes the form

$$\begin{aligned}
 \Pi(R) = & 2 \Pi_0 \frac{\exp(-ik_0 |\vec{d}|)}{|\vec{d}|} \\
 & - (ik_0 / 2\pi\eta_0) \int_{S_e} \int \frac{\exp(-ik_0 |\vec{w}|)}{|\vec{w}|} Z(W) \Pi(W) dS. \quad (11)
 \end{aligned}$$

Note that the first term under the integral in (8) is zero because the z-derivative of $[\exp(-ik_0 |\vec{w}|)/|\vec{w}|]$ is zero on a flat surface (S_e).

Equation (11) is the general integral equation for ground-wave propagation over a plane surface whose electrical properties vary as $Z(W)$.

In the case of a homogeneous surface, that is $Z(W)$ constant, we may assume the following form for Π :

$$\Pi(W) = 2 \Pi_0 \frac{\exp(-ik_0 |\vec{d} - \vec{w}|)}{|\vec{d} - \vec{w}|} F(W, Z), \quad (12)$$

where Π is a composite of the field due to a vertical electric dipole over an infinitely conducting plane surface and the attenuation, $F(W, Z)$, up to the point W due to the actual plane surface of impedance Z .

The substitution of this into (11) produces an equation which is the usual form for the Sommerfeld equation for ground-wave propagation over a plane homogeneous earth, namely,

$$\begin{aligned} F(R, Z) = & 1 - (ik_0 Z / 2\pi\eta_0) |\vec{d}| \exp(i k_0 |\vec{d}|) \\ & \times \int_{S_e} \int \frac{\exp(-ik_0(|\vec{w}| + |\vec{d} - \vec{w}|))}{|\vec{w}| \cdot |\vec{d} - \vec{w}|} \\ & \times F(W, Z) dS. \end{aligned} \quad (13)$$

In the inhomogeneous case, that is, $Z(W)$ not constant, we could replace $F(W, Z)$ in (12) by $\tilde{F}(W, Z(W))$ and obtain an equation similar to (13) above with $Z(W)$ under the integral. In the case where the surface S_e is divided into two regions S_e' and S_e'' (refer to fig. 3) with values of $Z(W)$ of Z and Z_1 , respectively, it is more convenient to proceed as follows. Supposing S_e' to be the largest of the two regions and S_e'' only a perturbation (an "island"), we can replace (6) by

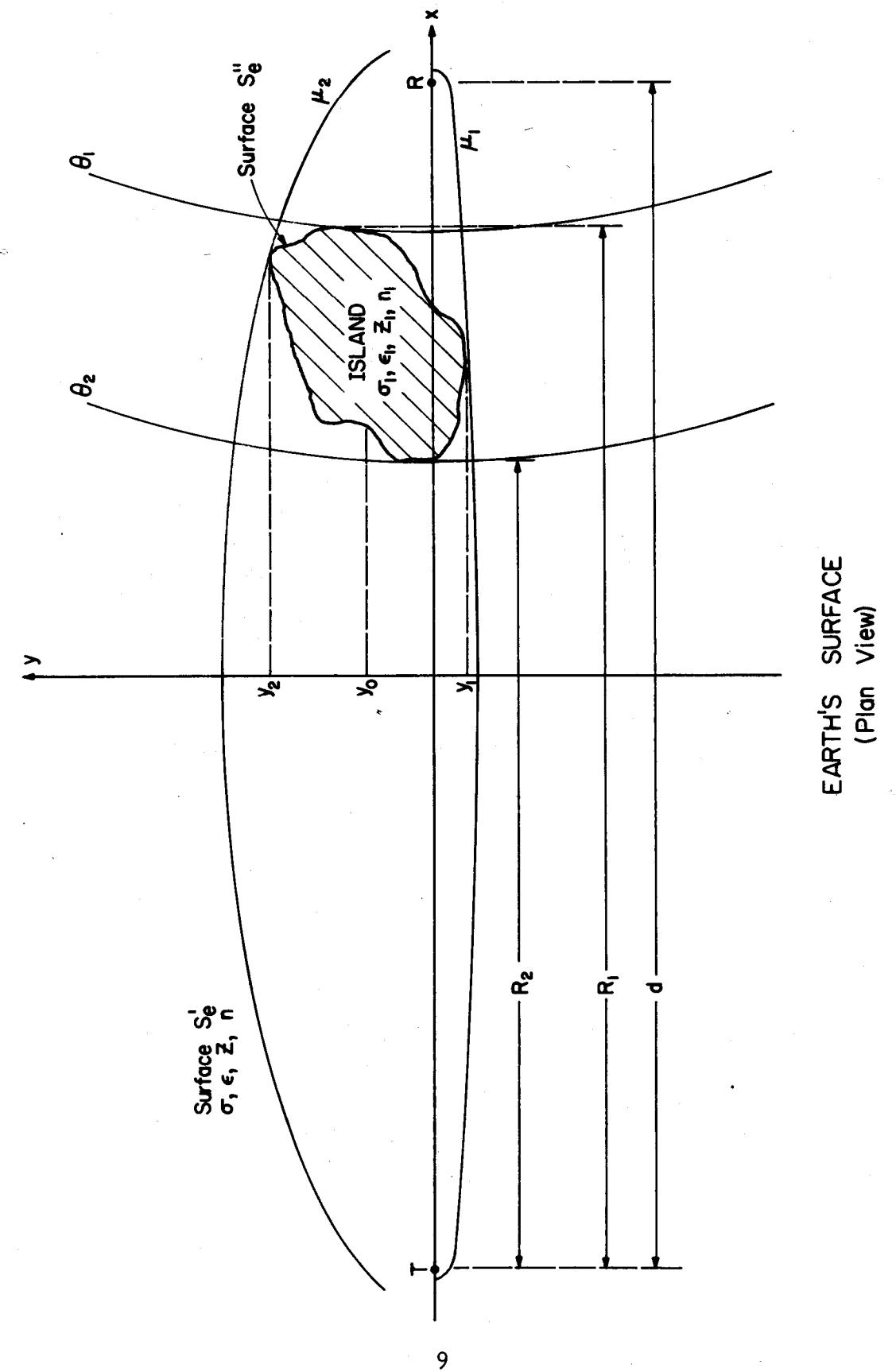


Figure 3. Top view of the path geometry.

$$\vec{\Pi}(W \text{ near } T) \cong \Pi_0 \frac{\exp(-ik_0 |\vec{d}|)}{|\vec{d}|} F(R, Z) \hat{z}, \quad (14)$$

(12) by

$$\Pi(W) = 2\Pi_0 \frac{\exp(-ik_0 |\vec{d} - \vec{w}|)}{|\vec{d} - \vec{w}|} F^*(W, Z, Z_1), \quad (15)$$

$Z(W)$ by $(Z - Z_1)$, and S_0 by S_0'' , proceed as above, and obtain the following result:

$$\begin{aligned} F^*(R, Z, Z_1) &= F(R, Z) - [ik_0(Z - Z_1)/2\pi\eta_0] |\vec{d}| \exp(ik_0 |\vec{d}|) \\ &\times \int \int_{S_0''} \frac{\exp(-ik_0(|\vec{w}| + |\vec{d} - \vec{w}|))}{|\vec{w}| \cdot |\vec{d} - \vec{w}|} \\ &\times F(W, Z) F^*(W, Z, Z_1) dS. \end{aligned} \quad (16)$$

If one now transforms this equation into elliptical coordinates, (μ, θ) , and applies the method of steepest descent to evaluate the μ -integral, (16) becomes (King, 1965; Tsukamoto et al., 1966) for an "island" inhomogeneity

$$\begin{aligned} F^*(d, Z, Z_1) &= F(d, Z) - (K(\mu_1, \mu_2)(Z - Z_1)/2\eta_0) \left(\frac{id}{\lambda}\right)^{\frac{1}{2}} \\ &\times \int_{\theta_1}^{\theta_2} F^*\left(\frac{1}{2}d(\cosh \mu_0 + \cos \theta), Z, Z_1\right) \\ &\times F\left(\frac{1}{2}d(\cosh \mu_0 - \cos \theta), Z\right) \\ &\times (\sinh^2 \mu_0 - \sin^2 \theta) / (\cosh^2 \mu_0 - \cos^2 \theta) d\theta, \end{aligned} \quad (17)$$

where d is the distance between the transmitter and receiver, μ_1 , μ_2 , θ_1 , and θ_2 define the boundaries of the "island" and μ_0 its "center-line" in the elliptical coordinate system (see fig. 3), and,

$$K(\mu_1, \mu_2) = \operatorname{erfc}(\sqrt{i k_0 d/2} \mu_1) - \operatorname{erfc}(\sqrt{i k_0 d/2} \mu_2). \quad (18)$$

The "center-line" μ_0 refers to the center line of the steepest descent integral and in our case is the value of μ ($= \mu_0$) corresponding to the value of y ($= y_0$) on the "island" which is closest to the line \overline{TR} (the x -axis). In figure 3, therefore, y_0 should be zero. The y_0 -line in figure 3 is displaced from this value only for the sake of illustrating its presence and is thereby unfortunately somewhat misleading. As a further matter of practical use of the model, the "steepest-descent-center-line" role of y_0 implies that R_1 and R_2 should be chosen as the "front" and "rear" edges of the "island" along the y_0 -line when \overline{TR} crosses the "island." If \overline{TR} does not cross the "island," the largest and smallest values of R (projected onto the x -axis) should be used.

Equation (17) must now be solved to obtain the solution to the inhomogeneous case. An iterative technique can be used to obtain a numerical solution. An initial approximation for F^* is substituted into the right side of (17) and this whole expression is then evaluated. From the left side of (17), we see that this result is the next approximation to F^* and hence can itself be inserted on the right side. Iteration proceeds in this manner until the desired accuracy is attained. The convergence of this method (the Neumann method) has been studied extensively (Lovitt, 1950, pp. 7, 110, 114); suffice it to remark (King, 1965) that it converges in our case. Following Tsukamoto et al. (1966), we chose as the initial approximation to F^* , the attenuation function for perpendicular propagation across a straight boundary, namely,

$$\begin{aligned}
F_o^*(d, Z, Z_1) = & F(d, Z) + ((Z - Z_1)/\eta_o) \left(\frac{id}{\lambda} \right)^{\frac{1}{2}} \\
& \times \int_0^{\theta_2} F \left(\frac{1}{2} d(\cosh h \mu_o + \cos \theta), Z \right) \\
& \times F \left(\frac{1}{2} d(\cosh h \mu_o - \cos \theta), Z_1 \right) d\theta , \quad (19)
\end{aligned}$$

where F is the Sommerfeld attenuation function and is given by

$$F(r, Z) \approx 1 - i\sqrt{\pi p} \exp(-p) \operatorname{erfc}(i\sqrt{p}) , \quad (20)$$

with

$$p = -(ik_o r/2) (Z/\eta_o)^2 , \quad (21)$$

and

$$\operatorname{erfc}(x) = (2/\sqrt{\pi}) \int_x^{\infty} e^{-t^2} dt . \quad (22)$$

The results presented in section 7 of this report were obtained from a computer program that calculates F and F^* by implementing the method of solution described in the paragraph above. The program is an adapted and modified version of a program used by King (1965) in his dissertation, and a much more complete description of the program and the problem can be found there.